Lecture 14

We may now establish additivity of $\int_X f d\mu$ on L^+. In fact:

Thm 2. If $\{f_n\}_{n=1}^{\infty}$ is a seq. in L^+ and $f = \sum_{n=1}^{\infty} f_n$, then

$$\int f = \sum_{n=1}^{\infty} \int f_n .$$

Pf. First, let's establish finite additivity. Take $f_1, f_2 \in L^+$ and recall $\exists$ increasing sequences of simple functions φ_n, ψ_n in L^+ s.t. $\varphi_n \nearrow f_1$, $\psi_n \nearrow f_2$ for all $x \in X$, and hence $\xi_n = \varphi_n + \psi_n$ is simple fcn s.t. $\xi_n \nearrow f_1 + f_2$. By Monotone Conv. Thm,

$$\int f = \lim_{n\to\infty} \int \xi_n \underset{\substack{\text{add. for}\\ \text{simple fcns}}}{=} \lim_{n\to\infty} \int \varphi_n + \lim_{n\to\infty} \int \psi_n = \int f_1 + \int f_2 .$$

By induction, we have

$$\int \sum_{n=1}^{N} f_n = \sum_{n=1}^{N} \int f_n .$$

Since $\sum_{n=1}^{N} f_n \nearrow f$ as $N \to \infty$, MCT $\Rightarrow$

$$\int f = \sum_{n=1}^{\infty} \int f_n \text{ as claimed. } \boxtimes$$

The following seems trivial but is important.

<u>Thm3</u> If $f \in L^+$, then

$$\int f \, d\mu = 0 \iff f = 0 \ \underbrace{\mu\text{-a.e.}}$$

except on a set $N \in \mathcal{M}$ s.t. $\mu(N) = 0$.

<u>Pf.</u> $\Leftarrow$: $f \le \tilde{f} = \begin{cases} 0, & x \notin N \\ \infty, & x \in N \end{cases} \Rightarrow$

$0 \le \int f \le \int \tilde{f}$. Also, $\varphi_n = n \cdot \chi_N \nearrow \tilde{f}$ and

$\int \varphi_n = 0 \Rightarrow \int \tilde{f} = 0$.

$\Rightarrow$: Consider $E_n = \{x : f(x) \ge \frac{1}{n}\}$.

Then $E_1 \subseteq E_2 \subseteq \ldots$ and $E = \bigcup_{n=1}^{\infty} E_n =$

$\{x : f(x) > 0\}$.

By def. $\varphi_n = \frac{1}{n}\chi_{E_n} \le f \Rightarrow \frac{1}{n}\mu(E_n) \le \int f = 0$.

Since $\mu(E) = \lim_{n\to\infty} \mu(E_n)$ by cont from below, we conclude $\mu(E)=0$, i.e. $f=0$ μ-a.e. $\boxtimes$

<u>Cor 1.</u> Let $\{f_n\}_{n=1}^{\infty}$ be seq. in L^+ s.t. $f_1 \le f_2 \le \dots$ μ-a.e. and $f_n \to f \in L^+$ μ-a.e. Then $\int f = \lim_{n\to\infty} \int f_n$.

<u>Pf.</u> By assumption, $\exists N \in \mathcal{M}$ s.t. $\mu(N)=0$ and $f_1 \le f_2 \le \dots$ and $f_n \to f$ for all $x \in X \setminus N$. Then, $\tilde{f}_n = f_n \cdot \chi_{X\setminus N}$ is increasing and converges to $\tilde{f} = f \cdot \chi_{X\setminus N}$. Thus, by MCT $\int \tilde{f}_n \to \int \tilde{f}$. By Thm 3, $\int \tilde{f}_n = \int f_n$ and $\int \tilde{f} = \int f$. $\boxtimes$

If monotonicity in MCT is dropped, the conclusion $\int f_n \to \int f$ may fail.

Ex 1. Consider $(X=(0,1), m)$, $f_n = n \cdot \chi_{(0,1/n)}$ and $f=0$. Then, $f_n(x) \to f(x)$ for all $x \in (0,1)$ but $\int f_n = n \cdot \frac{1}{n} = 1$ and $\int f = 0$.

Fatou's Lemma Assume $\{f_n\}_{n=1}^{\infty} \subseteq L^+$. Then

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

Pf. Let $\tilde{f}_N = \inf_{n \ge N} \{f_n\}$. Then $\tilde{f}_N \in L^+$, $\tilde{f}_1 \le \tilde{f}_2 \le \ldots$ and $f := \liminf_{n\to\infty} f_n = \lim_{N\to\infty} \tilde{f}_N$. Thus, by MCT

$$\int f = \lim_{N\to\infty} \int \tilde{f}_N .$$

On the other hand, $\tilde{f}_N \le f_N \Rightarrow$ $\int \tilde{f}_N \le \int f_N$. Taking liminf we conclude

$$\int f \le \liminf_{n \to \infty} \int f_n$$

as claimed. $\boxtimes$